INDIAN INSTITUTE OF ENGINEERING SCIENCE AND TECHNOLOGY, SHIBPUR

DEPARTMENT OF MATHEMATICS

GST SHEET

FIRST SEMESTER (For all Engineering Branches)

SUBJECT: MATHEMATICS-I

SUBJECT CODE: MA 101

SERIAL NUMBER 1:
FUNCTIONS OF SINGLE REAL VARIABLE AND SEVERAL REAL VARIABLES

1. If $u = sinax + cosax$, show that

$$u_n = a^n \{ 1 + (-1)^n \sin 2ax \}^{\frac{1}{2}}.$$

2. If $xsin\,\theta + ycos\,\theta = a\ and\ xcos\,\theta - ysin\,\theta = b$, prove that

$$\frac{d^p x}{d\theta^p} \cdot \frac{d^q y}{d\theta^q} - \frac{d^q x}{d\theta^q} \cdot \frac{d^p y}{d\theta^p}$$

is constant.

3. Prove that if $y = (x^2 - 1)^n$, then $(x^2 - 1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0.$ Hence show that if $z = D^n (x^2 - 1)^n$, then z satisfies the following second order differential equation:

$$(1 - x^2)\frac{d^2 z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0.$$

4. Let $P_n = D^n (x^n \, logx)$. Prove the recurrence relation

$$P_n = nP_{n-1} + (n-1)!.$$

Hence show that $P_n = n! \left(logx + 1 + \frac{1}{2} + \frac{1}{3} + \ ... \ + \frac{1}{n} \right).$

5. If $y = sin^{-1}x$, then show that

$(i)(1 - x^2)y_2 - xy_1 = 0;$

$(ii)(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0 .$

Find also the value of $(y_n)_0$.

$\boldsymbol{Ans)\ 0\ or\ \{1.3.5.\dots(n-2)\}^2 according\ as\ n\ is\ even\ or\ odd.}$

6. If $y = cosh(sin^{-1}x)$, prove that

$(i)(1 - x^2)y_2 - xy_1 - y = 0;$
$(ii)(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} = (n^2+1)\, y_n .$

7. Prove, by mathematical induction, that $\frac{d^n}{dx^n}\left(x^{n-1} e^{\frac{1}{x}}\right) = (-1)^n \frac{e^{\frac{1}{x}}}{x^{n+1}}$.

8. If $f(x) = tanx$, prove that

$$f^n(0) - n_{c_2}\ f^{n-2}(0) + n_{c_4}\ f^{n-4}(0) - \cdots = \sin\frac{n\pi}{2}.$$

9. If $y = x^{n-1} logx$, show that $y_n = \frac{(n-1)!}{x}$.

10. If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(\theta h), 0 < \theta < 1$, find θ when h = 7 and $f(x) = \frac{1}{1+x}$.

Ans) 1/7.

11. Show that

$$(x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + \frac{3}{2} x^{\frac{1}{2}} h + \frac{3}{2}.\frac{1}{2}\frac{h^2}{2!} \frac{1}{\sqrt{(x+\theta h)}},0 < \theta < 1.$$

Find θ, when x=0.

Ans) 9/64.

12. (i) Prove that $\lim_{h\to 0} \frac{f(a+h)-f(a-h)}{2h} = f'(a)$, provided f '(x) is continuous.

(ii) Prove that $\lim_{h\to 0} \frac{f(a+h)-2f(a)+f(a-h)}{h^2} = f''(a)$, provided $f''(x)$ is continuous.

13. Show that $\frac{x}{1+x} < \log(1+x) < x$ for all x > 0.

14. If $a = -1, b \geq 1\ and\ f(x) = 1/\mid x \mid$, prove that Lagrange's MVT is not applicable for f in [a, b]. But check that the conclusion of the theorem is TRUE if $b > 1+\sqrt{2}$.

15. Given: $y = f(x) = \frac{1}{\sqrt{1+2x}}$,

(i) Prove that $(1+2x)y_{n+1} + (2n+1)y_n = 0$.

(ii) Expand f(x) by Maclaurin's Theorem with remainder after n terms. Write the remainders both in Lagrange's and Cauchy's forms.

Ans) $\mathbf{f(x) = 1 - x + \frac{1.\ 3}{2!}x^2 - \frac{1.\ 3.5}{3!}x^3 + \cdots + (-1)^{n-1} \frac{1.\ 3.\ 5....(2n-3)}{(n-1)!}x^{n-1} + R_n}$,

where $\mathbf{R_n}$ **= Remainder after n terms in Lagrange's form**

$$= \frac{x^n}{n!}(-1)^n \frac{1.3.5....(2n-1)}{(1+2\theta x)^{\frac{2n+1}{2}}} (0 < \theta < 1),$$

$\mathbf{R_n}$ **= Remainder after n terms in Cauchy's form**

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1}(-1)^n \frac{1.3.5....(2n-1)}{(1+2\theta x)^{\frac{2n+1}{2}}} (0 < \theta < 1)$$

16. Expand the following functions in powers of x in infinite series stating in each case the conditions under which the expansion is valid:

(i) $cosx$, (ii) e^x, (iii) $log(1+x)$.

Ans) (i) $\mathbf{cosx = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \text{ to } \propto, \text{for all values of x.}}$

$\mathbf{(ii) e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \text{ to } \propto, \text{for all values of x.}}$

$\mathbf{(iii) log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \text{ to } \propto, \text{is valid for} -1 < x \leq 1.}$

17. If $y = e^{asin^{-1}x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$, prove that

(i) $(1-x^2)y_2 = xy_1 + a^2$

(ii) $(n+1)(n+2)a_{n+2} = (n^2 + a^2)\, a_n$,

and hence obtain the expansion of $e^{asin^{-1}x}$.

Ans) $\mathbf{1 + ax + \frac{a^2x^2}{2!} + \frac{a(a^2+1^2)}{3!} x^3 + \frac{a^2(a^2+2^2)}{4!} x^4 + \frac{a(a^2+1^2)(a^2+3^2)}{5!} x^5 + \cdots}$

18. Expand $(sin^{-1} x)^2$ in a series of ascending powers of x.

Ans) $\mathbf{\frac{1}{2!}.2x^2 + \frac{2^2}{4!}.2x^4 + \frac{2^2.\,4^2}{6!}.2x^6 + \frac{2^2.\,4^2.6^2}{8!}.2x^8 + \cdots}$

19. If $y = sin\,(m\, sin^{-1}x)$, show that

$$(1-x^2)y_{n+2} - (2n+1)\, xy_{n+1} + (m^2 - n^2)\, y_n = 0,$$

and hence obtain the expansion of $sin(msin^{-1}x)$.

Ans) $\mathbf{mx - \frac{m(m^2-1^2)}{3!}x^3 + \frac{m(m^2-1^2)(m^2-3^2)}{5!}x^5 - \cdots}$

20. $If\ y = e^{ax}\, cos\, bx$, prove that

$$y_2 - 2ay_1 + (a^2 + b^2)y = 0,$$

and hence obtain the expansion of $e^{ax}\, cos\, bx$.

Deduce the expansions of e^{ax} and $\cos bx$.

$$\textbf{Ans)}\ 1 + ax + \frac{a^2-b^2}{2!}x^2 + \frac{a(a^2-3b^2)}{3!}x^3 + \cdots,$$

$$e^{ax} = 1 + ax + \frac{a^2x^2}{2!} + \cdots,$$

$$\cos bx = 1 - \frac{b^2x^2}{2!} + \frac{b^4x^4}{4!} - \cdots$$

21. Show that the radius of curvature at the point (r, θ) on the cardioide $r = a(1 - \cos\theta)$ varies as √r.
22. Find the radius of curvature at the origin of the curve $x^3 + y^3 = 3axy$.

Ans) $3a/2, 3a/2$.

23. Show that for the ellipse $x^2/a^2 + y^2/b^2 = 1$, the radius of curvature at an extremity of the major axis is equal to half the latus-rectum.

24. If ρ_1 and ρ_2 be the radii of curvature at the ends of a focal chord of the parabola $y^2 = 4ax$, then show that

$$\rho_1^{-\frac{2}{3}} + \rho_2^{-\frac{2}{3}} = (2a)^{-\frac{2}{3}}.$$

25. Show that the radius of curvature at any point of the equiangular spiral subtends

a right angle at the pole.

26. Find the radius of curvature at the point (p, r) of the curve $r^{m+1} = a^m p$.

Ans) $\frac{a^m}{(m+1)r^{m-1}}$.

27. Show that the chord of curvature through the pole for the curve $p = f(r)$ is given by$2f(r) / f'(r)$.

28. Find the asymptotes of the curve

$$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$$

Ans) $6y - 6x + 7 = 0, 2y - 6x + 3 = 0, 6y + 3x + 5 = 0$.

29. Find the asymptotes of the curve

$$(y + x + 1)(y + 2x + 2)(y + 3x + 3)(y - x) + x^2 + y^2 - 8 = 0.$$

$$\boldsymbol{Ans)\ y + x + 1 = 0, y + 2x + 2 = 0, y + 3x + 3 = 0, y - x = 0.}$$

30. Find the asymptotes of the curve

$$x^2 (x + y)(x - y)^2 + 2x^3 (x - y) - 4y^3 = 0.$$

$$\boldsymbol{Ans)\ x = 2, x = -2, x - y + 2 = 0, x - y - 1 = 0, x + y + 1 = 0.}$$

31. Show that the asymptotes of the curve

$$x^2y^2 - a^2(x^2 + y^2) - a^3 (x + y) + a^4 = 0.$$

form a square two of whose angular points lie on the curve.

32. Find the equation of the cubic which has the same asymptotes as the curve

$$x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0.$$

and which touches the axis of y at the origin and goes through the point (3,2).

Ans) $\boldsymbol{x^3 - 6x^2y + 11xy^2 - 6y^3 - x = 0.}$

33. If $u = log\,(x^3 + y^3 + z^3 - 3xyz)$, show that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$,

(ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -\frac{3}{(x+y+z)^2}$.

34. Verify Euler's theorem for the function $u = \frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{5}}+y^{\frac{1}{5}}}$.

35. The side a of a triangle ABC is calculated from b, c, A. If there be small errors db, dc, dA in the measured values of b, c, A, show that the error in the calculated value of a is given by

$$da = cosC\,db + cosB\,dc + b\,sinC\,dA$$

36. If z be a differentiable function of x and y and if

$$x = c \cosh u \cos v, y = c \sinh u \sin v,$$

then prove that

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = \frac{1}{2} c^2 \ (\cosh 2u - \cos 2v) \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right).$$

37. If F(u, v) is a twice differentiable function of u, v and if $u = x^2 - y^2$ and v = 2xy, prove that

$$4(u^2 + v^2) \frac{\partial^2 F}{\partial u \, \partial v} + 2u \frac{\partial F}{\partial v} + 2v \frac{\partial F}{\partial u} = xy \left(\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2} \right) + (x^2 - y^2) \frac{\partial^2 F}{\partial x \, \partial y}.$$

38. A differentiable function f(x, y), when expressed in terms of the new variables u and v defined by

$$x = \frac{1}{2} (u + v), y = \sqrt{(uv)}$$

becomes g(u, v); prove that

$$\frac{\partial^2 g}{\partial u \, \partial v} = \frac{1}{4} \left(\frac{\partial^2 f}{\partial x^2} + 2 \frac{x}{y} \frac{\partial^2 f}{\partial x \, \partial y} + \frac{\partial^2 f}{\partial y^2} + \frac{1}{y} \frac{\partial f}{\partial y} \right).$$

39. Given that F is a differentiable function of x and y and that

$$x = e^u + e^{-v}, y = e^v + e^{-u},$$

prove that

$$\frac{\partial^2 F}{\partial u^2} - 2 \frac{\partial^2 F}{\partial u \, \partial v} + \frac{\partial^2 F}{\partial v^2} = x^2 \frac{\partial^2 F}{\partial x^2} - 2xy \frac{\partial^2 F}{\partial x \, \partial y} + y^2 \frac{\partial^2 F}{\partial y^2} + x \frac{\partial F}{\partial y} + y \frac{\partial F}{\partial y}.$$

40. If $f(x, y) = \begin{Bmatrix} xy \left(\frac{x^2 - y^2}{x^2 + y^2} \right), \text{when } (x, y) \neq (0,0) \\ 0 \text{, when } (x, y) = (0,0) \end{Bmatrix},$

show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

41. If $(x, y) = \begin{cases} \left(\frac{xy}{x^2+y^2}\right), \text{when } (x, y) \neq (0,0) \\ 0, \text{when } (x, y) = (0,0) \end{cases}$,

Show that both the partial derivatives f_x and f_y exist at (0, 0) but the function is

not continuous thereat.

42. Show that the function $f(x, y) = \sqrt{|xy|}$ is not differentiable at $(0,0)$, but that f_x and f_y both exist at the origin and have the value 0.

43. Show that the expansion of $f(x, y) = \sin xy$ in powers of $(x-1)$ and $(y-\frac{\pi}{2})$ upto and including second degree terms is

$$1 - \frac{1}{8}\pi^2(x-1)^2 - \frac{1}{2}\pi(x-1)\left(y - \frac{\pi}{2}\right) - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2.$$

44. Examine for maximum and minimum values of

$$f(x, y, z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z.$$

Ans) The given function has five stationary points (0,3,1), (0,1,-1), (1,2,0),(2,1,1),(2,3,-1).

The function is neither maximum nor minimum at (0,3,1), (0,1,-1), (2,1,1),(2,3,-1).

The function is minimum at (1, 2, 0).

45. Prove that the volume of the greatest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$.

SERIAL NUMBER 2:

INFINITE SERIES

1. Test the convergence of the series

$$\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \ldots$$

[Ans : **Divergent**]

2. Test the convergence of the series

$$1 + \frac{1}{2}.\frac{1}{3} + \frac{1.3}{2.4}.\frac{1}{5} + \frac{1.3.5}{2.4.6}.\frac{1}{7} + \ldots$$

[Ans: **Convergent**]

3. Test the convergence of the series

$$\frac{5}{1.2.4}+\frac{7}{2.3.5}+\frac{9}{3.4.6}+\frac{11}{4.5.7}+\dots$$ [Ans: **Convergent**]

4. Examine the convergence of the series

$$1+\frac{2^2}{3^2}+\frac{2^2.4^2}{3^2.5^2}+\frac{2^2.4^2.6^2}{3^2.5^2.7^2}+\dots$$ [Ans: **Divergent**]

5. Examine the convergence of the series

$$\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\dots.+\left(\frac{n}{2n+1}\right)^n+\dots$$ [Ans: **Convergent**]

6. Examine the convergence of the series

$$1+\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^4}+\dots$$ [Ans**: Convergent**]

7. Test the convergence of the series

$$\sum_{n=1}^{\infty}\left\{\frac{2.4.6.8\dots.2n}{3.5.7.9\dots.(2n+1)}\right\}^2.$$ [Ans: **Divergent**]

8. Using Comparison test prove that the series $\sum_{n=1}^{\infty} e^{-n^2}$ and $\sum_{n=1}^{\infty}\sin^3\left(\frac{1}{n}\right)$ both converges.

9. The series $\sum_{n=1}^{\infty}\frac{n}{2^n}$ converges by using Ratio test.

10. Investigate the convergence of the series:

$$1+\frac{1}{2^3}+\frac{1}{2^2}+\frac{1}{2^5}+\frac{1}{2^4}+\cdots.$$ [Ans**: Convergent**]

11. The series $1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\frac{1}{5^p}+\cdots$ converges for $p>1$ and diverges for $p\le 1$.

12. Test the convergence of the series $\sum_{n=1}^{\infty}\frac{1}{n}$. [Ans: **Divergent**]

13. If $\sum_{n=1}^{\infty}u_n$ be convergent series of positive real numbers prove that $\sum_{n=1}^{\infty}u_n^2$ is convergent.

14. Prove that the series $\frac{a}{b}+\frac{a(a+c)}{b(b+c)}+\frac{a(a+c)(a+2c)}{b(b+c)(b+2c)}+\dots\dots\dots\dots, a,b,c>0$ is convergent if $b>a+c$ and $b\le a+c$.

15. Test the convergence of the series:

$$\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{8}\right)+\tan\left(\frac{\pi}{12}\right)+\ldots\ldots\ldots\ldots\ldots\ldots$$ [Ans: **Convergent**]

16. Test the convergence of the series $\sum_{n=1}^{\infty} u_n$, where $u_n=\sqrt{n^4+1}-n^2$. [Ans: **Convergent**]

17. Investigate the convergence of the series:

$$\frac{1}{\log 2}+\frac{1}{\log 3}+\frac{1}{\log 4}+\cdots.$$ [Ans: **Divergent]**

18. Examine the convergence of $\frac{1^2}{2}+\frac{2^2}{2^2}+\frac{3^2}{2^3}+\frac{4^2}{2^4}+\ldots\ldots.$ [Ans: **Convergent**]

19. Using comparison test examine the convergence

$$\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\ldots\ldots\ldots+\frac{1}{(2n-1)(2n)}+\ldots\ldots$$ [Ans**. Convergent]**

20. Examine the convergence by D'Alembert's principle

$$\frac{4}{1!}+\frac{4^2}{2!}+\frac{4^2}{3!}+\frac{4^2}{4!}+\ldots\ldots\ldots\ldots+\frac{4^2}{n!}+\ldots$$ [Ans**. Convergent]**

21. Examine the convergence of the series $\sum u_n$, where $u_n=\frac{2^n}{(n+1)^n}$. [Ans**. Convergent]**

22. Examine the convergence of $\sum_{n=1}^{\infty}\frac{n^n}{n!}$. [Ans**. Divergent]**

23. Examine the convergence of the series $\sum u_n$, where $u_n=\frac{x^n}{n!}$ $(x>0)$. [Ans**. Convergent]**

24 . Examine the convergence of the series $\sum u_n$, where $u_n=\frac{\sqrt{n}}{n^3+1}$. [Ans**. Convergent]**

25. Apply Cauchy's root test check the convergence of the series

$$\frac{1+2}{2.1}+\left(\frac{2+2}{2.2}\right)^2+\left(\frac{3+2}{2.3}\right)^3+\ldots\ldots\ldots+\left(\frac{n+2}{2.n}\right)^n+\ldots\ldots$$ [Ans**. Convergent]**

26. Apply Cauchy's root test check the convergence of the series

$$\frac{1}{2}+\frac{1}{3^2}+\frac{1}{4^3}+.......+\frac{1}{(n+1)^n}+.....$$ [Ans**. Convergent]**

27. Determine the radius of convergence of the power series $\sum_{n=1}^{\infty}\frac{x^{3n}}{2^n}$ **. [Ans**. $\sqrt[3]{2}$]

28. Determine the radius of convergence of the power series $\sum_{n=1}^{\infty}\frac{n!(x+2)^n}{n^n}$. **[Ans**.1/e]

29. Determine the radius of convergence of the power series $x+\frac{x^2}{2^2}+\frac{x^3}{3^3}+\frac{x^4}{4^4}+...$ **[Ans**. 1]

30. Test the convergence of the series

$\sum_{n=1}^{\infty}Cos(\frac{1}{n})$ [Ans**. Divergent]**

SERIAL NUMBER 3:

MULTIPLE INTEGRALS

1. Evaluate $\iint_R\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy$, where R consists of points in the positive quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$ **[Ans:** $\frac{\pi ab}{8}$ **]**
2. Evaluate $\iint_E\left(x^2+y^2\right)dx\,dy$, over the region E bounded by $xy=1,\ y=0, y=x, x=2.$ **[Ans:** $\frac{47}{24}$ **]**
3. Evaluate $\iint\sqrt{(4x^2-y^2)}\ dx\,dy$, over the triangle formed by the straight lines $y=0, x=1, y=x.$ **[Ans:** $\frac{\sqrt{3}}{6}+\frac{\pi}{9}$ **]**
4. Show that $\iiint_E\left(a^2b^2c^2-b^2c^2x^2-c^2a^2y^2-a^2b^2z^2\right)^{1/2}dx\,dy\,dz=\frac{\pi^2}{4}a^2b^2c^2$, where E is the region bounded by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

5. Show that $\iiint\limits_E \frac{dx\,dy\,dz}{(1+x+y+z)^3} = \frac{1}{2}\left(\ln 2 - \frac{5}{8}\right)$, where E is the tetrahedron bounded by the planes $x=0, y=0, z=0, x+y+z=1.$

6. By changing the order of integration, prove that

$$\int_0^1 dx \int_0^{(1-x^2)^{1/2}} \frac{dy}{(1+e^y)\sqrt{1-x^2-y^2}} = \frac{\pi}{2}\ln\left(\frac{2e}{1+e}\right).$$

7. Find the area of that part of the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ which lies between the coordinate planes.

[Ans: $\frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}$ **]**

8. Prove that the area of the surface of the sphere $x^2+y^2+z^2=9$ is 36π.

9. Prove that the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4}{3}\pi abc.$

10. Evaluate $\iint xy(x+y)\,dx\,dy$ over the region bounded by $y=x^2$ and $y=x$. **[Ans:** $\frac{3}{56}$ **]**

11. Prove that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\,dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx$. Does the double integral $\iint\limits_E \frac{(x-y)}{(x+y)^3}\,dx\,dy$ exists if $E=[0,1;0,1]$? . Justify your answer.

12. Express $\int_0^{\frac{\pi}{2}} dx \int_0^{\cos x} x^2 dy$ as a double integral and evaluate it. [Ans. $(-8+\pi^2)/4$]

13. Show that $\iint\limits_D e^{\frac{y}{x}}\,dx\,dy$ where D is the triangle bounded by $y=x,\ y=0$ and $x=1$ is $\frac{(e-1)}{2}$.

14. Show that $\iint x^{\frac{1}{2}} y^{\frac{1}{3}} (1-x-y)^{\frac{2}{3}}\,dx\,dy$ over the triangle bounded by

$$x=0, y=0 \text{ and } x+y=1 \text{ is } \frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}.$$

15. Show that $\iint e^{\frac{y-x}{x+y}}\,dx\,dy$ taken over the triangle with vertices at $(0,0)$, $(0,1)$ and $(1,0)$ is $\frac{1}{4}\left(e-\frac{1}{e}\right)$.

16. Find the area bounded by one arch of the cycloid $x=a(\theta-\sin\theta),\ y=a(1-\cos\theta)$ and the x-axis.

17. Find the volume of the solid generated by revolving the cardioide $r = a(1 - \cos\theta)$ about the initial line.

18. Evaluate $\iint (x^2 + y^2)dxdy$ over the region in the positive quadrant for which $x + y \le 1.$ **[Ans.** $\frac{1}{6}$ **]**

19. Evaluate $\iint xy(x+y)dxdy$ over the area bounded by $y = x^2$ and $y = x.$

[Ans. $\frac{a^3}{3}\log(\sqrt{2}+1)$ **]**

20. Evaluate by suitable transformations , $\iint (x+y)^2 dxdy$ over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ **[Ans.** $\frac{\pi ab(a^2+b^2)}{4}$ **]**

21. Evaluate by suitable transformations, $\iint x^2 ydxdy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ **[Ans.** $\frac{a^3b^2}{15}$ **]**

22. Evaluate by suitable transformations, $\iint r^2 \sin\theta drd\theta$ over the upper half of the circle $r = 2a\cos\theta.$ **[Ans.** $\frac{2a^3}{3}$ **]**

23. Evaluate $\iint xydxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$. **[Ans.** $\frac{a^4}{8}$ **]**

24. Evaluate $\int_0^1 dy \int_y^1 e^{-x^2} \mathrm{dx}$ **[Ans.** $(-1+e)/2e$ **]**

25. Evaluate $\int_0^{\pi/2}\int_0^{\pi} \cos(x+y)dxdy.$ **[Ans. -2]**

26. Evaluate: $\iint_R y\,dx\,x\,dy$ where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y.$ **[Ans.** $\frac{48}{5}$ **]**

27. Evaluate $\iint (a^2 - x^2 - y^2)\,dx\,dy$ over the semi circle $x^2 + y^2 = ax$ in the first quadrant. **[Ans.** $\frac{5\pi a^4}{64}$ **]**

28. Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\,dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx.$

29. Evaluate $\iint \sqrt{\left[\frac{1-(x^2/a^2)-(y^2/b^2)}{1+(x^2/a^2)+(y^2/b^2)}\right]}\,dx\,dy$ over the positive quadrant of the ellipse $(x^2/a^2)+(y^2/b^2)=1.$ **[Ans.** $\frac{1}{4}ab\pi(\frac{1}{4}\pi - 1)$ **]**

30. By using the transformations $x + y = u,\ y = uv,$ show that $\int_{x=0}^{1}\int_{y=0}^{1-x} e^{\frac{y}{x+y}}\,dy\,dx = \frac{e-1}{2}.$

31. Transform the integral $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy$ by changing to polar coordinates and hence evaluate it. **[Ans** $\frac{a^4}{4}$ **]**

32. Evaluate $\iint \sqrt{(4 - x^2 - y^2)}\,dx\,dy$ over the region bounded by the semicircle $x^2 + y^2 - 2x = 0$ lying in the first quadrant. **[Ans.** $\frac{4}{3}(\pi - \frac{4}{3})$ **]**

33. Evaluate $\iint \left[\frac{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}{1 + \frac{x^2}{a^2} + \frac{y^2}{b^2}}\right]^{1/2} dx\,dy$ over the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$

[Ans. $ab\frac{\pi(\pi-2)}{8}$ **]**

34. Change the order of integration in $\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate this. [**Ans.** $\frac{3}{8}$ **]**

35. Fin the area between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$. **[Ans.** $\frac{16a^2}{3}$ **]**

36. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. **[Ans.** πab **]**

37. Fin the volume of the region bounded by the surface $y = x^2$ and $x = y^2$ and the planes $z = 0, z = 3$. **[Ans. 1]**

38. Find the volume of the tours generated by revolving the circle $x^2 + y^2 = 4$ about the line $x = 3$. **[Ans.** $24\pi^2$ **]**

39. Evaluate the integral $\iiint x\,y\,z\,dz\,dy\,dx$ over the volume enclosed by three co-ordinate planes and the plane $x + y + z = 1$. **[Ans.** $\frac{7}{21600}$ **]**

SERIAL NUMBER 4:
IMPROPER INTEGRALS

1. What do you mean by proper and improper integrals ? Explain the concept of convergence of improper integrals in the following three cases:
   (i) improper integrals of type-I, (ii) improper integrals of type-II, (iii) improper integrals of mixed type (or, type –III).

2. Evaluate each of the following improper integrals, if exists:
   (i) $\int_0^\infty \frac{dx}{x^2+a^2}$, $a > 0$ (ii) $\int_1^\infty \frac{dx}{x^2}$ (iii) $\int_0^\infty e^{-ax} dx$, $a > 0$ (iv) $\int_{-\infty}^\infty xe^{-x^2} dx$

   (v) $\int_a^\infty \cos(bx) dx$ (vi) $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ (vii) $\int_1^2 \frac{dx}{(x+1)\sqrt{x^2-1}}$ (viii) $\int_0^1 \frac{dx}{x}$

   [Ans.: (i) $\frac{\pi}{2}$ (ii) 1 (iii) 1/a (iv) 0 (v) Non-convergent (oscillatory) (vi) $\frac{\pi}{2}$ (vii) $1/\sqrt{3}$ (viii) diverges to ∞]

3. Show that

(i) the improper integral $\int_a^\infty \frac{dx}{x^\mu}$ (a>0) converges for $\mu > 1$ *and diverges for* $\mu \leq 1$

(ii) the improper integral $\int_a^b \frac{dx}{(x-a)^\mu}$ converges for $0 < \mu < 1$ *and diverges when* $\mu \geq 1$.

What happens when $\mu \leq 0$?

4. What do you mean by Cauchy principal value of (i) an improper integrals of type-I and (ii) an improper integrals of type-II.

5. Find the Cauchy principal values of (i) $\int_{-1}^1 \frac{dx}{x^3}$ and (ii) $\int_{-\infty}^\infty \frac{dx}{x^2+a^2}$, if exist.

[Ans. (i) 0 (ii) π]

6. Justify yourself with example that existence of Cauchy principal value never ensures the convergence of an improper integral in the general sense.

7. What do you mean by absolutely convergent and conditionally convergent improper integrals. Give example in each case.

8. Examine the nature of convergence of the following improper integrals:

(i) $\int_0^\infty \frac{x^2dx}{3x^4+20}$ (ii) $\int_0^\infty \frac{xdx}{\sqrt{x^4+x^2+2}}$ (iii) $\int_1^\infty \frac{sin^2x}{x^2}\,dx$ (iv) $\int_1^\infty \frac{\log x}{x^2}\,dx$

(v) $\int_0^\infty \frac{x^2dx}{3x^{5/2}+20}$ (vi) $\int_0^\infty e^{-x^2}dx$ (vii) $\int_0^1 \frac{dx}{(1+x)\sqrt{x}}$ (viii) $\int_0^1 \frac{\log x\,dx}{\sqrt{x}}$

(ix) $\int_0^1 \frac{\sqrt{x}dx}{\log x}$ (x) $\int_0^\infty \frac{\cos x\,dx}{\sqrt{x^5+x^2+2}}$ (xi) $\int_0^1 \frac{\log x}{1+x}\,dx$ (xii) $\int_1^\infty \frac{dx}{x\log x}\,dx$

[Ans. (i) convergent (ii) divergent (iii) convergent (iv) convergent (v) divergent
(vi) convergent (vii) convergent (viii) convergent (ix) divergent (x) convergent
(xi) convergent (xii) divergent]

9. Prove that the Gamma function (or, the Second Eulerian Integral), defined by

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx,$$

is convergent for n > 0.

10. Prove that the Beta function (or, the First Eulerian Integral), defined by

$$\mathrm{B}(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,$$

is convergent for m > 0, n > 0.

11. Verify the following relations involving Gamma function:

(i) $\int_0^\infty e^{-ax} x^{n-1} dx = \Gamma(n)/a^n$, a > 0, n > 0.

(ii) $\Gamma(n+1) = n\,\Gamma(n)$, n > 0

(iii) $\Gamma(1) = 1$

(iv) $\Gamma(n+1) = n!\,, when\ n\ is\ a\ positive\ integer.$

12. Verify the following relations involving Beta function:

(i) B(m, n) = B(n, m), m, n > 0

(ii) $\mathrm{B}(m, n) = \int_0^\infty x^{m-1}/(1+x)^{m+n} dx = \int_0^\infty x^{n-1}/(1+x)^{m+n} dx,$ m, n > 0

(iii) B(m,n) = $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx,$ m, n > 0

(iv) B(m, n) = $2\int_0^{\pi/2} sin^{2m-1}\theta \ cos^{2n-1}\theta \ d\theta,$ m, n > 0

(v) B(1/2, 1/2) = π

13. Show that the relation between Beta and Gamma functions is given by

B(m, n) = $\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, m > 0, n > 0 .

Hence deduce that

(i) $\Gamma(1/2) = \sqrt{\pi}$

(ii) $\int_0^{\pi/2} sin^p\theta\, cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$, p, q > -1.

14. Prove the Duplication formula

$$2^{2m-1}\,\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2m), m > 0.$$

15. Evaluate the following integrals:

(i) $\int_0^{\infty} e^{-x^2} dx$ (Ans. $\sqrt{\pi}/2$)

(ii) $\int_0^{\pi/2} sin^4\theta\, cos^5\theta\, d\theta$ (Ans. 8/315)

(iii) $\int_0^{\pi/2} sin^9\theta\, d\theta$ (Ans. 128/315)

(iv) $\int_0^{\pi/2} cos^4\theta\, d\theta$ (Ans. $3\pi/16$)

(v) $\int_0^{\infty} a^{-x^2} dx\, , a > 0$ (Ans. $\frac{\sqrt{\pi}}{2\sqrt{\log a}}$)

(vi) $\int_0^{\infty} e^{-4x} x^{3/2} dx$ (Ans. $\frac{3\sqrt{\pi}}{128}$)

(vii) $\int_0^1 x^3(1-x^2)^{5/2} dx$ (Ans. 2/63)

(viii) $\int_0^1 x^a(1-x^b)^n dx\, , a, b, n > 0$ (Ans. $\frac{B\left(\frac{a+1}{b}, n+1\right)}{b}$)

(ix) $\int_0^1 \frac{dx}{(1-x^6)^{1/6}}$ (Ans. $\pi/3$)

(x) $\int_0^{\infty} e^{-x^4} dx\, .\int_0^{\infty} e^{-x^4} x^2 dx$ (Ans. $\pi/8\sqrt{2}$)

SERIAL NUMBER 5

ORDINARY DIFFERENTIAL EQUATIONS

1. **Solve the following differential equations :**

a) $y'' - y' - 2y = 6e^{3x}$ (here prime denotes differentiation with respect to variable)

b) $y'' - y' - 2y = 6e^{2x}$

c) $y'' - y' = 6e^{x}$

d) $y'' - y' - 2y = 6e^{-x}$

e) $y'' - 2y' + y = 6e^{x}$

f) $2y'' - y' - 3y = e^{(3/2)x}$

g) $(D^2 + 2D + 3)y = 2x^2,\; D \equiv d/dx$

h) $(D^2+2D+1)y=2x^2$

i) $(D^2-1)y=2x^2$

j) $(D^2-2D+1)y=e^x+2x$

k) $(D^2+3D+5)y=xe^{2x}$

l) $(D^2-3D+2)y=x^2e^{2x}$

m) $(D^3-D^2-2D)y=0$

n) $(D^2-4)y=\sin x$

o) $(D^2+9)y=\sin 3x$

p) $(D^2-2D+1)y=x\cos x$

q) $(D^2+D+1)y=x\cos x$

r) $(D^2-1)y=e^x\sin(x/2)$

s) $(D^2+1)y=\sin x\sin 2x$

t) $(D^2+5D+6)y=\cosh 2x$

u) $(D^2+D+2)y=x\sinh x$

v) $(D^2-1)y=xe^x\sin x$

w) $(D^2+4)y=x\sin^2 x$

x) $(D^2+3D+5)y=\sin x\cos 2x$

y) $(D^2+3D)y=\cos 5x$

z) $(D^2+D)y=x$

2. Solve the following differential equations :

a) $(x^2D^2-xD+1)y=\ln x$

b) $(x^2D^2+xD+1)y=\sin(\ln x^2)$

c) $(x^2D^2-4xD+6)y=x$

d) $(x^2D^2+xD-1)y=\sin(\ln x)+x\cos(\ln x)$

e) $(x^2D^2-4xD+6)y=x^2$

f) $(x^2D^2-2xD-4)y=x^2+2\ln x$

g) $(x^2D^2+xD+1)y=\ln x\sin(\ln x)$

3. Using method of variation of parameters solve the following equations:

a) $y''+y=\sec x$

b) $y''+2y'+y=x^2e^{2x}$

c) $(1+x)y'' + xy' - y = (1+x)^2$, given that $y = x$ *and* $y = e^{-x}$ are two solutions of the corresponding homogeneous equation.

d) $x^2y'' + xy' - y = x^2e^x$, given that $y = x$ *and* $y = 1/x$ are two solutions of the corresponding homogeneous equation.

e) $(x-1)y'' - xy' + y = (x-1)^2$, given that $y = x$ *and* $y = e^x$ are two solutions of the corresponding homogeneous equation.

f) $y'' + 4y = \tan 2x$

g) $y'' - 2y' + y = e^x \ln x$

h) $y'' + 2y' + y = e^{-x} \ln x$

i) $y'' + y = \operatorname{cosec} x$

j) $y'' - y = xe^x$

4. Solve the following initial value problems :

a) $y'' + 16y = 17e^x, \quad y(0) = 6, \; y'(0) = -2$

b) $y'' - 3y' + 2y = 10\sin x, \quad y(0) = 1, \quad y'(0) = -6$

c) $y'' - 4y' + 3y = 10\sin x, \quad y(0) = 2, \quad y'(0) = -1$

d) $y'' + 4y' + (4 + w^2)y = 0, \quad y(0) = 1, \quad y'(0) = w - 2$

e) $y''' - y'' - y' + y = 0, \quad y(0) = 2, \quad y'(0) = 1, \quad y''(0) = 0$

5. Solve in series the following differential equations in the neighbourhood of the origin (an ordinary point):

a) $y'' + 2x^2y = 0$

b) $y'' - xy' + y = 0$

c) $y'' + y = 0$

d) $y' - y = 0$

e) $y' - 2xy = 0$

6. **Legendre Differential Equation** : $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

Bessel Differential Equation : $x^2y'' + xy' + (x^2 - \nu^2)y = 0$

When **n is an integer**, one solution of Legendre Differential Equation is

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

where M = n/2 or (n-1)/2, whichever is integer, called **Legendre Polynomial of degree n.**

One solution of Bessel Differential Equation (ν is integer or not) is

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{\nu+2m}}{2^{2m+\nu} m!\, \Gamma(\nu+m+1)},$$

called **Bessel function of the first kind or simply Bessel function order ν.** When ν is an integer say n , it is written as $J_n(x)$.

Rodrigue's Formula : $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$

Generating Function : $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x)$

$$e^{\frac{x}{2}(t - \frac{1}{t})} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

7. Prove that $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0,\ when\ m \neq n\ and\ \frac{2}{2n+1}\ when\ m = n$
8. Prove the following Recurrence relations :
   a) $\frac{d}{dx}[x^{\nu} J_{\nu}(x)] = x^{\nu} J_{\nu-1}(x)$
   b) $\frac{d}{dx}[x^{-\nu} J_{\nu}(x)] = -x^{-\nu} J_{\nu+1}(x)$
   c) $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x} J_{\nu}(x)$
   d) $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J'_{\nu}(x)$
   e) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$
   f) $nP_n(x) = xP'_n(x) - P'_{n-1}(x)$
   g) $(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$
   h) $P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x)$